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Shape Dynamics and Birkhoff's Theorem

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Abstract

In this report, we followed a series of papers based on a recently developed theory of gravity called shape dynamics. We explored the construction principles behind linking theories that allow symmetry trading. Finally, we reproduced Birkhoff's theorem in shape dynamics, following a recent paper. It is shown that shape dynamics possess different solutions to general relativity when a constant mean extrinsic curvature gauge can not be found.

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Chapter 1

Introduction

In the last century, there has been increasing speculation that angles comprise the fundamental quantities of nature and not length. Theories embracing this idea, or conformal theories, have long been sought after particularly in gravity as a new approach to the major challenges encountered in realising a quantum theory of gravity. Conformal theories lead naturally to the idea that geometry or in some sense 'shape', underlie nature and this is particularly appealing in obtaining a more fundamental description of reality.

Recently, a new conformal theory of gravity has been developed called shape dynamics [1] and is the subject of this report. It has the potential to resolve some of the issues in quantum gravity, for example, the problem of time, but is in fact part of a larger framework based on relational principles [2]. Core to shape dynamics is Mach's principle which states that dynamics should admit a description which depends only on relative quantities. To understand these principles better as well as how shape dynamics can be considered a Machian theory but without giving a historical account beginning with Newton, we discuss this for point-particles. This is easiest to visualise and captures the essence of what shape dynamics is about.

1.1 Relational Principles

We often think of the positions of particles as being located in 3-dimensional Euclidean space. In shape dynamics, the background space is irrelevant – only those quantities which relate particles have physical meaning e.g. the inter-particle separations. If we consider three particles, from a relational point of view, we can rotate, translate and rescale arbitrarily without changing the physical picture; the absence of an absolute background means there is no distinction between these configurations. Geometrically it is not hard to imagine how three particles form a triangle. Rescaling preserves the angles between sides; global rotations and global translations have no physical effect since the notion of orientation is lost without an absolute background to frame it against. Configuration space is replaced by the so-called shape space – the space of all distinct *shapes*. Instead of the evolution of 'positions' we now have the dynamics of shapes, or, *shape dynamics*.

These notions are more abstract in the gravitational setting but the principles remain much the same. When we go over to fields, shape space is replaced by the space of all possible geometries or 3-metrics that describe the spatial geometry of the Universe at different 'instants'. Curves in this space describe the evolution of geometries and the group of rotations, translations and rescalings is replaced by the group of angle-preserving transformations and diffeomorphisms.[1]

What are these 'instants'? In shape dynamics time is not fundamental but emerges from the theory from considering physical changes that is, without a change in shape, the idea of time is lost. These instants are then distinct configurations or geometries. One might ask: How does one formulate dynamics without time? Time is of course a very entrenched notion in our daily lives, physics no exception. Motivated by Mach's principle, English relationalist Julian Barbour, conceived a method called best-matching for precisely this task of deriving dynamics of point particles in the absence of absolute structure [2]. Most importantly, this technique allows a description of dynamics without resorting to a concept of time.

In developing the general framework, one is naturally lead to the need for an action principle. The redundancy in configuration space (given by rotations etc) means it is best formulated in terms of fibre bundles – the language of gauge theories. But, continuing with the heuristic discussion so far, we will only touch on the key points and how this relates to shape dynamics.

Ideally the action principle is formulated on shape space, however, a theory of shape does not admit a simple mathematical description [3]. We are led to formulate it on the configuration space. Consequently, we are obliged to handle the redundancies that arise in configuration space as opposed to working in the space of physical degrees of freedom. Physical states can be described in more than one way, in fact, there will be a family of such states (forming an equivalence class) for which states are related to each other by rotations, translations, and rescalings. It is not surprising then that a relational action principle must allow for many different curves in configuration space, however, in shape space there will be a single curve.[2] In the usual principle of least action, the end-points of all possible trajectories are fixed. This is not so for a configuration space that admits many equivalent configurations. End-points are allowed to vary at both ends but we must choose a gauge, i.e., we must fix a size and an orientation in configuration space – this is unfortunately the price we pay in a space with absolute structures. From some simple arguments[3] it can be shown that the momenta p associated with the 'generalised positions' of the rotations, translations and dilatations must vanish at all points along the dynamical trajectory: $p = 0$. This is a constraint on allowed solutions.

These types of constraints form the basis of the formalism to follow, so an understanding of how such constraints can arise in general and its geometrical interpretation will be needed before we proceed to shape dynamics. We will make explicit mention of the steps which are crucial to the construction of shape dynamics along the way.

1.2 Shape Dynamics as a Theory of Gravity

Shape dynamics holds that the physical gravitational degrees of freedom are, not surprisingly, related to shape [4]. There are 6 degrees of freedom of the 3-metric but only two are physical. It is well established that the diffeomorphism symmetry is responsible for three of these but the last non-physical degree of freedom is crucial difference between general relativity and shape dynamics. In general relativity this degree of freedom is due to the fact that simultaneity is relative. Mathematically, this means we have a freedom in what 'time' parameter we choose. On other hand, shape dynamics regards this last non-physical degree

[1] Roughly speaking diffeomorphisms are coordinate transformations.

[2] This curve is the 'projection' of this equivalence class onto shape space.

[3] See appendix A.2.

of freedom to be the freedom to rescale shapes in three dimensions. Both theories agree on the number of physical degrees of freedom but disagree on the symmetries they should possess. Lengths determined by 3-metrics, $\mathbf{g}$, can be rescaled by a factor $\Omega > 0 : \mathbf{g} \rightarrow \Omega\,\mathbf{g}$ while still preserving angle

$$\cos\theta = \frac{\mathbf{g}(\delta x, \delta y)}{\sqrt{\mathbf{g}(\delta x, \delta x)}\sqrt{\mathbf{g}(\delta y, \delta y)}}.$$

These basic notions form the foundation of shape dynamics as we will now demonstrate.

1.2.1 Outline of Research

In this report we investigate shape dynamics as a theory 'dual' to general relativity. We construct shape dynamics by explicit calculation using *linking theories* for both open and closed manifolds and comment briefly on the equivalence of the two theories. We show following [5] one result from shape dynamics, namely Birkhoff's theorem and how the solution differs from general relativity. In the next section we introduce the formalism that will be used widely throughout this report. In section 3 we outline the general method for symmetry trading. In section 4 we apply this method to general relativity to obtain shape dynamics for both the open and closed manifold case.

Notation

We use signature $-+++$ and the Einstein summation convention (repeated indices are summed over). Since we are primarily working in the 3+1 description of general relativity, tensor spatial components will be denoted by latin indices $a, b, c, \ldots$ and go from 1 to 3. E.g., g_{ab} denotes the components of the (spatial) 3-metric. The determinant of the 3-metric will be denoted g and the covariant derivative associated with the 3-metric will be denoted ∇_a. To simplify notation, we will often denote the set of generalised coordinates by q^i, e.g., For N degrees of freedom, $f(q^i, p_i)$ denotes a function of the the canonical variables $q^1, \ldots, q^N, p_1, \ldots, p_N$. Finally, we use units $c = 1$

Chapter 2

Background

2.1 Constrained Hamiltonian Formalism

The nature of relational dynamics means there are redundancies in the Hamiltonian formulation. In shape dynamics this is dealt with using Dirac's Hamiltonian formulation for constrained systems [6]. The redundancies manifest in algebraic constraints, depending on position and momenta. These constraints are an integral part of gauge theories as Hamiltonian systems and will be encountered often in the development of shape dynamics.

2.1.1 Constrained Hamiltonian systems

In the standard formulation of Lagrangian mechanics one extremises the action $\int \mathrm{d}t \mathcal{L}(q^i, \dot{q}^i)$ to obtain the equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i}$$

or equivalently

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^j \partial \dot{q}^i}\ddot{q}^j = -\frac{\partial^2 \mathcal{L}}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial \mathcal{L}}{\partial q^i}$$

where $\partial^2 \mathcal{L}/\partial \dot{q}^j \dot{q}^i =: W_{ij}$ is the Hessian. For many systems (not encountered when one first learns analytical mechanics) W_{ij} is not invertible and so the acceleration, $\ddot{q}^i$, will not be uniquely determined. Consequently, for a given system, trajectories in configuration space will not be unique; there exist many trajectories satisfying the Euler-Lagrange equations arising from the same point in configuration space.

When we pass to the Hamiltonian framework, we define the momenta conjugate to q^i by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

and invert to obtain $\dot{q}^i$ in terms of the canonical pairs (q^i, p_i). If we are unable to do this then the Jacobian $\frac{\partial p_i}{\partial \dot{q}^j}(= W_{ij})$ will be singular – precisely the condition we found in the Lagrangian framework. Consequently, not all momenta are independent of each other, but there exist relations between q's and p's:

$$\phi_m(q^{\cdot} p_i) = 0,$$

the so called constraints which are central to gauge theories as Hamiltonian systems, which we frequently encounter throughout this report.

The space of allowed states will not be all of the phase space manifold Γ, but a submanifold for which the constraints hold called the constraint surface. The allowed states on the constraint surface are given by the set

$$\{(q^i, p_i) \in \Gamma : \ \phi_m(q^i, p_i) = 0, \ \ m = 1, \dots, M\} \tag{2.1}$$

However we can extend the definition ϕ_m off this submanifold so long as the constraints are satisfied on it. It will become convenient to introduce some terminology now and so we will say a function, $f(q,p)$ is *weakly zero* if on the submanifold it is equal to zero but off it can take on non-zero values. We denote this by $f(q,p) \approx 0$. A function that is *strongly zero* has the familiar meaning that it is zero everywhere.

In dealing with constraints (typically holonomic) the usual approach is through a judicious choice of coordinates but they can also be dealt with by adding onto the Lagrangian (holonomic or not) a linear combination of ϕ_m:

$$\mathcal{L} \to \mathcal{L} + \sum_m u^m \phi_m$$

where $u^m = u^m(t)$ are the arbitrary Lagrange multipliers. The general Hamiltonian is

$$\mathcal{H}_{\text{can}}(q,p) + \sum_m u^m \phi_m \tag{2.2}$$

where $\mathcal{H}_{\text{can}}$ is the canonical Hamiltonian and so the action becomes

$$S = \int \mathrm{d}t \ \left(p_i \dot{q}^i - \mathcal{H}_{\text{can}}(q^i, p_i) - u^m \phi_m \right) .$$

The equations of motion are still obtained from stationary S (i.e. $\delta S = 0$) for fixed endpoints. Now we have two dynamical equations and one equation enforcing the constraints ϕ_m:

$$\dot{q}^n = \frac{\partial \mathcal{H}}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \tag{2.3}$$

$$\dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n}, \tag{2.4}$$

$$\phi_m = 0. \tag{2.5}$$

It is important that the constraints are not evaluated before differentiating otherwise the equations of motion would just reduce to the unconstrained case. With the Poisson bracket, $[\cdot, \cdot]$, defined on phase space as usual, the equations of motion can be written in the form

$$\dot{g} = [g, \mathcal{H}] + \sum_m u^m [g, \phi_m]. \tag{2.6}$$

(This differs from the usual case because we have additional terms.) At this point, the Lagrange multipliers are arbitrary but may need to be fixed in order that the dynamics lead to physical solutions. However we can see that if they are unspecified (i.e. unfixed), then we have an element of arbitrariness in the dynamical trajectory of g. These particular constraints with unspecified Lagrange multipliers are the origin of the arbitrariness and we see how transformations can arise which do not change our physical state (see §2.1.6).

Note, in the construction of shape dynamics, the canonical Hamiltonian is zero so the first Poisson bracket vanishes and we are left with the 'evolution' generated by constraints. This an indication that the theory is parametrisation invariant, i.e., we can change the labelling parameter without consequence.

2.1.2 Geometric picture

It will be useful to have a geometric understanding of Hamiltonian mechanics when treating constraints so we will introduce some concepts related to the symplectic structure of phase space.

The symplectic structure of phase space naturally defines a 2-form $\boldsymbol{\omega} = \mathrm{d}q^i \wedge \mathrm{d}p_i$ (which we can think of as a 'metric') that assigns a unique vector field V_G to each phase space function G through

$$G \mapsto V_G \quad \text{s.t.} \quad \boldsymbol{\omega}(V_G, \cdot) = \mathrm{d}G.$$

Here d is the exterior derivative and we contract $\boldsymbol{\omega}$ with the vector field V_G. In particular, if $G = \mathcal{H}$ then in local coordinates, $\mathrm{d}\mathcal{H} = (\partial \mathcal{H}/\partial q^i)\,\mathrm{d}q^i + (\partial \mathcal{H}/\partial p_i)\,\mathrm{d}p_i$. Since $V_{\mathcal{H}} = \dot{q}^i \partial/\partial q^i + \dot{p}_i \partial/\partial p_i$, we have

$$\boldsymbol{\omega}(V_{\mathcal{H}}, \cdot) = \mathrm{d}q^i(V_{\mathcal{H}}) \wedge \mathrm{d}p_i - \mathrm{d}q^i \wedge \mathrm{d}p_i(V_{\mathcal{H}}) = \dot{q}^i\,\mathrm{d}p_i - \dot{p}_i\,\mathrm{d}q^i$$

and equating coefficients with $\mathrm{d}\mathcal{H}$ we obtain Hamilton's equations. Thus, in the geometrical picture, solving the canonical equations amounts to finding the integral curve corresponding to the vector field $V_{\mathcal{H}}$. Contracting on both slots:

$$\boldsymbol{\omega}(V_F, V_{\mathcal{H}}) = [F, \mathcal{H}],$$

which is of course the equation of motion for F. However, we can also consider the 'evolution' generated by functions other than the Hamiltonian. In general we have $[F, G] = \boldsymbol{\omega}(V_F, V_G) =: \delta F$ and this is interpreted as the change of F generated by the vector field V_G. We call G the generator and V_G the *Hamiltonian vector field.*

2.1.3 Dirac-Bergmann Algorithm

Once we have found all constraints $\{\phi_m : m = 1, \ldots, M\}$ we must ensure they are *propagated* along their Hamiltonian vector fields, i.e., constraints continue to hold at future times. This 'self-consistency condition' is the statement that $\dot{\phi}_k \approx 0$ and results in a system of M linear equations

$$[\phi_k, \mathcal{H}_{\text{can}}] + \sum_{m=1}^{M} u_m[\phi_k, \phi_m] \approx 0 \qquad k = 1, \ldots, M \tag{2.7}$$

to be solved for the hitherto arbitrary functions u^m.

There are multiple scenarios when solving this system. The most important case is when there is no unique solution. Recall from linear algebra that this implies, in this example, that the matrix $[\phi_k, \phi_m]$ is degenerate; the solution will involve a finite number of free parameters and consequently some u^m's will remain arbitrary. Such a solution is characteristic of gauge generators present in the constraint set $\{\phi_m\}$ (see next section). Each Lagrange multiplier that is not fixed implies non-physical degree of freedom in the system. A fixed u^m therefore restores this degree of freedom to a physical status and in the process eliminates the associated arbitrariness.

In the second case, we may find the u^m's drop out completely leaving us with additional relations $\phi'_n(q, p)$ further restricting the constraint surface, eq. (2.1). We add these to our constraint set: $\{\phi_m, \phi'_n\}$. There are other cases which can arise by considering the possibilities when solving a system of linear equations but these are by far the two most important cases.

We will now look at a particular characterisation of constraints which are, by construction, self-propagating.

2.1.4 First-Class and Second-Class Constraints

The set of all constraints is not unique, in that linear combinations of constraints in the set $[\phi_p]$ is also a constraint. However, the number of independent constraints does not change.

Particular combinations will now be introduced which have a certain property; it is just such a property that allows us identify them with gauge transformations. We call a constraint, χ_i, *first-class* if

$$[\chi_i,\, \phi_p] \approx 0$$

for each $p = 1, .., M$. A constraint ρ_k which is not first-class is called *second-class*, that is, if it has a non-vanishing Poisson bracket with at least one constraint. The total Hamiltonian may now be written as

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{can}} + \sum_j v^j \rho_j + \sum_k u^k \chi_k \tag{2.8}$$

and the equations of motion are then $\dot{g} = [g, \mathcal{H}_{\text{tot}}]$.[1] Notice that if a constraint is first-class then propagating it,

$$\dot{\chi}_i = \sum_j v^j [\chi_i, \chi_j] + \sum_k u^k [\chi_i, \rho_k] \approx 0$$

since first-class constraints Poisson commute with all other constraints. In the case of second-class constraints, it will be weakly non-zero. Demanding consistent dynamics[2], we obtain a system of linear equations $\rho'_k \approx 0$ for each k. Solving the system fixes the v^j leaving the u^k unspecified. This is the source of arbitrariness in evolution and is represented by the last term in eq. (2.8).

A system that possesses gauge symmetries contain first-class constraints. It is therefore desirable to identify all first-class constraints and hence *gauge generators*. This involves constructing first-class out of the set $\{\phi_m,\, \phi'_n\}$ by taking linear combinations: $\chi_i = \sum_j A_{ij} \phi_j$. The number of first-class constraints to be found is just the degeneracy of the matrix $[\phi_k, \phi_m] =: C_{km}$ i.e. the dimension of the solution space when solving eq. (2.7).

2.1.5 Symplectic View

Returning to the symplectic picture, we can see now that for first-class constraints

$$\chi'_j = [\chi_j, \chi_k] = \boldsymbol{\omega}(V_{\chi_j}, V_{\chi_k}) \approx 0,$$

where the prime denotes a small change in χ_j (analogous to $\dot{\chi}$). The small change in χ_j comes from the fact that the Poisson bracket

$$[\cdot, \chi_k] = \frac{\partial \chi_k}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial \chi_k}{\partial q^i} \frac{\partial}{\partial p_i},$$

has the properties of a derivation, and so acts as a directional derivative in the direction of the Hamiltonian vector field, V_{χ_k},. The vanishing of Poisson brackets implies that there is no change in the value of χ_j and so it is preserved along integral curves of V_{χ_k}, which is equivalent to saying that the *vector field is tangent to the constraint surface.* Thus we have a characterisation of first-class constraints in terms of its geometric properties.

In the case of second-class constraints, $\chi'_i \not\approx 0$, so that the induced vector field is not tangent to the constraint surface but take us off it as we have shown before. The non-physical

[1] Note $\mathcal{H}_{\text{can}} + \sum_j v^j \rho_j$ is a first-class function and is often denoted $\mathcal{H}'$.

[2] We will use the terms "consistent dynamics" and "propagating constraints" interchangably.

dynamics it generates needs to be handled in a way that does not interfere with the true dynamics. We deal with second-class constraints by 'projecting' the vector field back onto the first-class constraint surface by defining a generalised bracket called the Dirac bracket:

$$[\cdot,\cdot]_{\mathrm{DB}} = [\cdot,\cdot] - [\cdot,\rho_i](C^{-1})^{ij}[\rho_j,\cdot]$$

where $(C^{-1})^{ij}$ is the inverse of $C_{ij} = [\rho_i, \rho_j]$, an antisymmetric matrix constructed from the minimal set of independent second-class constraints. This bracket shares many of the same properties as the Poisson bracket and is identical for first-class arguments but crucially it is strongly equal to zero for second-class constraints. As a result the symplectic flow of second-class constraints is empty so $[\cdot, \rho_i] = 0$, and we can effectively set $\rho_i = 0$. We emphasise this non-trivial step is the basis on which we can perform the symmetry trading procedure (to be outlined).

2.1.6 Gauge Transformations

The evolution of a state is determined by the Hamiltonian but by now it has been realised there are arbitrary functions in $\mathcal{H}_{\mathrm{tot}}$ (2.8), implying many possible trajectories in phase space. This ambiguity of state can be reconciled with *gauge transformations* – the idea that we can represent the same physical state in multiple ways in phase space. (If this was not so we would reach the absurd conclusion that we could affect the physical evolution by writing down a different u^k.) Such transformations come from the fact that we have more mathematical structure than physical structure in our theory and consequently not all p's and q's are in reality observable (What is observable has been the subject of much debate but here we leave the interpretation vague.)

Each first-class constraint is tied to an arbitrary coefficient $u^k(t)$ (which we call the gauge) so the value of a dynamical variable will depend on the chosen gauge. By considering the difference in the value of a phase space function F at some fixed time, using two different gauges, we obtain

$$\delta F = \delta u[F, \chi_k] \tag{2.9}$$

(See [6, 7] for details.) This can not be a physically meaningful difference since we have merely changed from one arbitrary function to another. Thus, it is a (infinitesimal) gauge transformation

$$F \to F + \delta u[F, \chi_k]$$

with χ_k called the *gauge generator*.

Before we proceed, it should be noted that not all first-class constraints generate gauge transformations; in the case of canonical general relativity this would imply the Hamiltonian is a gauge generator and that the evolution is a gauge transformation. In fact, the gauge degrees of freedom are entangled with the physical degrees of freedom complicating the interpretation of the Hamiltonian [8].

Often it is useful to gauge-fix the system by imposing extra conditions (constraints) called *gauge-fixing conditions*, thereby eliminating the non-physical degrees of freedom and reformulating the theory on a reduced phase space. In the linking theory (section 3.1) this is the crucial step in obtaining shape dynamics.

Example

Suppose we have two free particles in 1-dimension with Hamiltonian

$$H_{\text{can}} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$$

and the constraint $\chi = p_1 + p_2 \approx 0$ (total canonical momentum zero). Propagating this constraint we find the identity $0 = 0$ so there are no other constraints. It is first-class since all 1-dimensional constraint algebras are first-class. The most general Hamiltonian is

$$H_{\text{tot}} = H_{\text{can}} + u(t)\chi$$

and the equations of motion are

$$\dot{q}^1 = \frac{p_1}{m_1} + u, \qquad \dot{p}_1 = 0$$

and

$$\dot{q}^2 = \frac{p_2}{m_2} + u, \qquad \dot{p}_2 = 0$$

Although the total canonical momentum is constrained to be zero, in another frame we have mechanical momentum

$$m_1\dot{q}^1 + m_2\dot{q}^2 = p_1 + p_2 + m_{\text{tot}}u \approx m_{\text{tot}}u$$

where $m_{\text{tot}} = m_1 + m_2$. This is the idea of a different gauge for different observers. Of course we can take $u = 0$ and have zero total momentum.

The solutions to these equations are

$$q^1(s) = \frac{p_1}{m_1}(s - s_0) + \left(\int_{s_0}^{s} u(s')\ \mathrm{d}s'\right) + q^1(s_0), \qquad p_1(s) = p_1(s_0)$$

and

$$q^2(s) = \frac{p_2}{m_2}(s - s_0) + \left(\int_{s_0}^{s} u(s')\ \mathrm{d}s'\right) + q^2(s_0), \qquad p_2(s) = p_2(s_0)$$

so

$$q^1(s) - q^2(s) = \frac{p_1}{m_1}(s - s_0) - \frac{p_2}{m_2}(s - s_0) + q^1(s_0) - q^2(s_0)$$

does not depend on the gauge, i.e., it is *gauge-invariant*. It is a physically meaningful quantity – the separation between two particles. gauge transformations

Chapter 3

Outline of Procedure

3.1 Linking Theory Formalism

In this section we will give a treatment of gauge theories in the constraint formalism described above and outline the general procedure of constructing a 'special' gauge theory called a *linking theory*. This linking theory has the necessary property that allows us to recover from it two equivalent gauge theories with different gauge symmetries.

The linking theory is constructed from an existing Hamiltonian theory (e.g. canonical general relativity) by enlarging its phase space by artificially introducing degrees of freedom. We call this phase space the *extended phase space* for obvious reasons. These degrees of freedom are non-physical, so we have not modified the theory but added some extra structure. The introduction of these auxiliary degrees of freedom comes at a cost: we must impose additional constraints – one for each degree of freedom introduced – to maintain the same physical degrees of freedom. In shape dynamics there is one extra constraint required corresponding to the conformal factor, ϕ.

How we gauge fix these degrees of freedom determines which gauge theory we recover. However, both theories can be brought into explicit equivalence by performing an appropriate total gauge fixing, hence the name linking theory. For example, in electromagnetism one has many gauges available in choosing A_μ, e.g., the Lorenz and Coulomb gauges. These are *partial* gauge fixing conditions, so distinguish two separate theories. Under a complete gauge fixing both theories are brought into explicit equivalence– this is the idea behind the general relativity and shape dynamics being considered equivalent. (Furthermore, this can be seen from the initial value problems coinciding [9].) However, in the partial gauge fixing both theories encode different gauge symmetries and this is the case we are interested in.

It should be noted that the following is a general procedure and methods presented here are not exclusive to shape dynamics but can be applied to other gauge theories such as electrodynamics [10].

3.1.1 Construction Principle

To simplify the discussion in this section, we will only consider systems with first-class constraints. (Second-class constraints can always be dealt with using the Dirac-bracket.) The canonical variables will be $\mathbf{q} = (q^1, \ldots, q^N)$ and $\mathbf{p} = (p_1, \ldots, p_N)$.

The construction principle is as follows.

- We enlarge the phase space of an 'initial' theory by introducing one auxiliary degree of freedom given by (ϕ, π). (This will be canonical general relativity in our case.)

- We impose a first-class constraint of the form $\pi \approx 0$ to render the auxiliary degree of freedom non-physical. This is analogous to the best-matching constraint (and the detail can be found in appendix A.2). This completes the enlargement.

- A canonical transformation is performed on all variables such that π transforms as

$$\pi - \pi_0(\phi, \mathbf{q}, \mathbf{p}).$$

- We assume the set of constraints can be split into two sets. In one set, the constraints will have the form

$$\phi - \phi_0(q, p),$$

  and the in second set, will contain all constraints that are first-class with respect to π. We denote these constraints by χ_j. The constraint set is

$$\pi - \pi_0(\phi, \mathbf{q}, \mathbf{p}) \approx 0 \tag{3.1}$$
$$\phi - \phi_0(\mathbf{q}, \mathbf{p}) \approx 0 \tag{3.2}$$
$$\chi_j(\phi, \mathbf{q}, \mathbf{p}) \approx 0 \tag{3.3}$$

- There are two *distinguished* gauge-fixings. These are $\phi \approx 0$ which gauge-fixes (3.1) to be second-class and $\pi \approx 0$ which gauge-fixes (3.2) to also be second class. Recall from the Dirac-Bergmann algorithm that this fixes the Lagrange multipliers associated with these gauge-fixed constraints.

- We apply each gauge-fixing in turn to determine separate gauge theories.

- The constraint system with the gauge-fixings included will no longer be first class with respect to each other. These constraints must be set to zero in order to obtain a system with consistent dynamics. Setting these constraints to zero is equivalent to setting π strongly equal to π_0 in (3.1) with the gauge-fixing $\phi \approx 0$, and setting ϕ strongly equal to ϕ_0 with the gauge-fixing $\pi \approx 0$.

- This completes the phase space reduction and depending on which gauge-fixing we have chosen, will determine which remaining constraints we have left over. Thus, we have effectively *traded symmetries.*

This is the generic symmetry trading procedure for gauge theories which we apply to general relativity. The crucial step is that there are two constraints that can be solved to give the form (3.1) and (3.2).

Chapter 4

Shape Dynamics

4.1 A Linking Theory for General Relativity

4.1.1 Hamiltonian Formalism

Shape dynamics is a Hamiltonian theory of gravity based on the Arnowitt-Deser-Misner (ADM) description of general relativity (often used in quantum gravity attempts and numerical relativity). In the ADM formulation, a globally hyperbolic spacetime is split into a stack of spatial hypersurfaces Σ_t each labeled by a different parameter t. How we choose to split or *foliate* spacetime reflects the relativity of simultaneity in GR. To each foliation there is an associated *lapse function*, $N(x)$, measuring the rate of change of proper time with respect to coordinate time of an observer moving between hypersurfaces Σ_t and $\Sigma_{t+\delta t}$, and, a *shift vector*, $N^a(x)$, measuring the spatial difference between two observers – one moving normal to Σ_t, the other moving along lines of constant spatial coordinate. The internal geometry of the hypersurface is described by the *intrinsic curvature* determined from the 3-metric g_{ab}, while the *extrinsic curvature* given by K_{ab} describes the geometry as an embedding in spacetime. In addition, there is a further freedom in the parameter t. This *reparametrisation invariance* is a non-dynamical symmetry wherein changing the labelling does not change the solution – only how it is 'traversed'. Theories with this feature are characterised by a zero Hamiltonian [7], as is the case here. The canonical variables are the 3-metric g_{ab} and the conjugate momentum p^{ab}. The total Hamiltonian is

$$\mathcal{H}_{\text{tot}} = \int H \,\mathrm{d}^3x = \int \left(N(x)\mathcal{H} + N_a(x)\mathcal{H}^a\right) \mathrm{d}^3x, \tag{4.1}$$

where N and N^a are the lapse and shift and

$$\mathcal{H}(x) = \frac{1}{\sqrt{g}}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) - \sqrt{g}R \tag{4.2}$$

$$\mathcal{H}^a(x) = -2\nabla_b p^{ab}. \tag{4.3}$$

Here p is the trace of the canonical momentum $p = g_{ab}p^{ab}$ and g is the determinant of the 3-metric. Notice that neither $\dot{N}$ nor $\dot{N}^a$ appear in $\mathcal{L} = \int \mathrm{d}^3x(p^{ab}\dot{g}_{ab} - H)$. Thus, by the Euler-Lagrange equations, $\delta\mathcal{L}/\delta N = \delta\mathcal{L}/\delta N^a = 0$ which therefore determines the ADM constraints

$$\mathcal{H}(x) \approx 0 \quad \text{and} \quad \mathcal{H}^a(x) \approx 0 \tag{4.4}$$

for each point $x \in \Sigma_t$. The constraints are then understood to be infinite dimensional with N and N^a playing the role of Lagrange multipliers.

Smearing

Since we will be working with (infinite-dimensional) fields we will make frequent use of a technique called *smearing*. The idea is that when taking functional derivatives we would like to avoid Dirac-delta distributions appearing in our equations and so we *smear* with arbitrary functions. For example, consider a field $\varphi(x)$ whose functional derivative with respect to itself is

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta(x,y).$$

Then in smeared form we would first integrate:

$$\varphi(f) := \int \mathrm{d}^3x\sqrt{g}f(x)\varphi(x) \tag{4.5}$$

and then take the functional derivative,

$$\frac{\delta\varphi(f)}{\delta\varphi(x)} = \int \mathrm{d}^3y\sqrt{g}f(y)\delta(x,y) = f(x),$$

for $f(x)$ arbitrary. Thus, instead of dealing with distributions, we have algebraic functions which we can easily manipulate.

ADM Constraints

Now that we have introduced smearing we show that the constraints (**??**) are the generators of symmetries. Both constraints form a first-class set and so, according to Dirac, are the generators of gauge transformations. The *diffeomorphism constraint*, $\mathcal{H}^a$, generates spatial diffeomorphisms:

$$[g_{ab}, \mathcal{H}^a(\xi_a)] = \pounds_{\boldsymbol{\xi}} g_{ab} =: \delta g_{ab} \tag{4.6}$$

that is, g_{ab} and $g_{ab} + \pounds_{\boldsymbol{\xi}} g_{ab}$ are two physically indistinguishable 3-geometries (the diffeomorphism symmetry). Here $\pounds_{\boldsymbol{\xi}}$ denotes the Lie derivative in the direction of the vector field $\boldsymbol{\xi}$. The *Hamiltonian constraint*, $\mathcal{H}$, does not have a straightforward interpretation; in shape dynamics it is not regarded only as the generator of gauge transformations but contains in it a part responsible for dynamics [11]. If this was considered a pure gauge generator then the total Hamiltonian eq. (4.1) would consist of only gauge generators in that we would cease to have physical dynamics. Instead, we would trace out all physically equivalent states in phase space; this curve is called the *gauge-orbit*.

4.2 Shape Dynamics: Closed Manifold Case

The simplest formulation of shape dynamics is of asymptotically flat geometries, i.e., open manifolds with spatial metric isomorphic to the Euclidean 3-metric at infinity [9,12]. However, the need to specify boundary conditions in the asymptotically flat case is against the spirit of a Machian theory, so instead we will present shape dynamics in its original formulation as a theory on compact manifolds without boundary [1].

In the following sections we will outline the construction of shape dynamics from the linking theory above and show how general relativity emerges from the same extended theory. We will also show how shape dynamics can be extended to the asymptotically flat case assuming certain boundary conditions.

Extended Phase Space

To implement conformal invariance we introduce a phase space variable $\phi(x)$ – the gauge parameter in the conformal transformation $g_{ab} \rightarrow e^{4\phi(x)} g_{ab}$ (the factor of 4 simplifies the form of the Ricci tensor). The conjugate momentum, $\pi(x)$ together with $\phi(x)$, form the conformal phase space Γ_{conf}. The conformal phase space together with the initial phase space is the extended phase space, $\Gamma_{\text{ex}} = \Gamma \times \Gamma_{\text{conf}}$. The inclusion of these auxiliary variables introduces additional degrees of freedom. To ensure we retain the same number of degrees of freedom we need to impose the first-class constraint

$$\mathcal{C} := \pi(x) \approx 0,$$

which removes the extra degree of freedom making Γ_{ex} a *gauge* extension. This constraint is analogous to point mechanics, where finds that the momentum conjugate to the group parameter vanishes for all trajectories, whether dynamical or not (appendix A.2). It is the generator of conformal transformations, as we will see.

An alternative view can be taken: the Lagrangian does not change under the enlargement to Γ_{ex} and consequently has no $\dot{\phi}(x)$ dependence. The momenta is thus $\pi = 0$, which we recognise as the constraint above. However, this is not just an embedding of Γ in Γ_{ex} since $\phi(x)$ may not be vanishing; $\phi(x)$ plays an important role in the local conformal invariance.

Volume-Preserving Conformal Transformations

Implementing conformal invariance under the full group of conformal transformations,

$$g_{ab} \rightarrow e^{4\phi} g_{ab}, \tag{4.7}$$

results in too strong a symmetry leading to 'frozen' dynamics [10]. Thus, we restrict to a subgroup of transformations which preserve volume only.[1] Consequently, not all parameters ϕ are permissible, only those which preserve volume,

$$V = \int_{\Sigma_t} \mathrm{d}^3x \sqrt{g},$$

are allowed and will be denoted $\hat{\phi}(x)$. (Henceforth we suppress the x dependency to simplify notation but it should be noted the conformal transformations are local.) To find this $\hat{\phi}$, observe that under arbitrary conformal transformations, the volume transforms as

$$V' = \int \mathrm{d}^3x \sqrt{|e^{4\phi} g_{ab}|} = \int \mathrm{d}^3x \, e^{6\phi} \sqrt{g},$$

where $|\cdot|$ denotes the determinant. Consider the subgroup of conformal transformations for which their parameters differ by a number c, i.e., $\hat{\phi} = \phi + c$ or $e^{\hat{\phi}} = e^c e^{\phi}$. Now, $(g_{ab}, \hat{\phi})$ and (g_{ab}, ϕ) are two systems identical up to a global scale factor e^c. We will now determine the unique number $c = c[\phi]$, for which given any ϕ, the transformation

$$g_{ab} \rightarrow e^{4\hat{\phi}} g_{ab}, \tag{4.8}$$

where $\hat{\phi} = \phi + c$, will leave the volume invariant.

[1] In the asymptotically flat case there is no such restriction and the formulation is much simpler.

To show this, we first define the spatial mean of a function, f

$$\langle f \rangle := V^{-1} \int_\Sigma \mathrm{d}^3x \sqrt{g} f(x).$$

The functional $\phi \mapsto c \in \mathbb{R}$ given by

$$c[\phi] = -\frac{1}{6} \ln \langle e^{6\phi} \rangle,$$

then gives the scalar

$$\hat{\phi} = \phi - \frac{1}{6} \ln \langle e^{6\phi} \rangle, \tag{4.9}$$

for which the volume is invariant. Indeed

$$V' = \int \mathrm{d}^3x \, e^{6\hat{\phi}} \sqrt{g} = V \langle e^{6\hat{\phi}} \rangle = V \left\langle \frac{e^{6\phi}}{\langle e^{6\phi} \rangle} \right\rangle = V, \tag{4.10}$$

i.e., the volume is preserved under conformal transformations $g_{ab} \to e^{4\hat{\phi}} g_{ab}$. We will see how this new symmetry will come at the cost of another symmetry, namely the refoliation invariance.

Implementing the Symmetry

We now make a canonical transformation T on the variables such that the metric transforms as eq. (4.8) and thereby artificially introducing the conformal symmetry. The following computations can be verified in appendix B.1.3.

The generating functional of this transformation is

$$F(g_{ab}, \phi; P^{ab}, \Pi) = \int \mathrm{d}^3x \left(e^{4\phi} g_{ab} P^{ab} + \phi \Pi \right),$$

where P^{ab}, Π denote the transformation of p^{ab} and π respectively. The canonical variables are transformed as follows:

$$\begin{aligned} g_{ab} &\to e^{4\hat{\phi}} g_{ab}, \\ p^{ab} &\to P^{ab} = e^{-4\hat{\phi}} \left(p^{ab} - \frac{1}{3} \langle p \rangle (1 - e^{6\hat{\phi}}) \sqrt{g} g^{ab} \right), \\ \pi &\to \Pi = \pi - 4(p - \langle p \rangle \sqrt{g}). \end{aligned}$$

where $p = g_{ab} p^{ab}$, and trivially $\phi \to \phi$. The diffeomorphism constraint (4.3) becomes

$$\mathcal{H}^a \to T\mathcal{H}^a = -2e^{-4\phi} (\nabla_b p^{ab} - 2(p - \langle p \rangle \sqrt{g}) \nabla^a \phi). \tag{4.11}$$

The constraint $\mathcal{C}$ becomes

$$T\mathcal{C} = \pi - 4(p - \langle p \rangle \sqrt{g}), \tag{4.12}$$

which we use to rewrite (4.11) as

$$T\mathcal{H}^a = -2\nabla_b p^{ab} + \pi \nabla^a \phi \approx 0. \tag{4.13}$$

These two constraints, (4.11) and (4.13), are identical to each other only on the constraint surface and differ off it by an overall scale factor $e^{-4\phi}$. This does not affect the analysis as the physical dynamics will always remain on the constraint surface.

The Hamiltonian constraint takes a more complicated form:

$$\mathcal{H} \quad \rightarrow \quad T\mathcal{H} = \frac{e^{-6\hat{\phi}}}{\sqrt{g}} \left(p^{ab}p_{ab} - \frac{1}{2}p^2 + \frac{1}{3}p\langle p\rangle\sqrt{g}(1 - e^{6\hat{\phi}}) - \frac{1}{6}\langle p\rangle^2 g(1 - e^{6\hat{\phi}})^2 \right) \sqrt{g} - e^{2\hat{\phi}}R' \tag{4.14}$$

where $R' = R - 8\nabla^2\phi - 8\nabla^a\phi\nabla_a\phi$. Since Poisson brackets are invariant under canonical transformations (see appendix A.1) all constraints remain first-class with respect to each other i.e. Poisson bracket vanishes in old and new variables.

We check that $T\mathcal{C}$ generates gauge transformations by first smearing it with an arbitrary scalar:

$$\delta g_{ab} = [g_{ab}, T\mathcal{C}(\sigma)] = (\sigma - \langle\sigma\rangle)g_{ab},$$

indeed, this is a volume-preserving conformal transformation so we have successfully implemented the conformal symmetry we were after.
Finally, the total Hamiltonian of this extended theory is the linear combination of the smeared constraints $T\mathcal{H}$, $T\mathcal{H}^a$, and $T\mathcal{C}$:

$$\mathcal{H}_{\text{LT}} = \int \mathrm{d}^3x \left(N\, T\mathcal{H} + \xi_a\, T\mathcal{H}^a + \rho\, T\mathcal{C}\right).$$

where N, ξ_a and ρ are Lagrange multipliers. The equations of motion are then given by

$$\dot{\mathcal{F}} = [\mathcal{F}, \mathcal{H}_{\text{LT}}].$$

In the next section we show that there are two partial gauge fixings:

$$\Sigma_1 = \{\phi(x) \approx 0, \forall x \in \Sigma\} \quad \text{and} \quad \Sigma_2 = \{\pi(x) \approx 0, \forall x \in \Sigma\},$$

that leads us to general relativity and shape dynamics respectively. Both gauge fixed theories are equivalent as they derive from the same linking theory but each contains different symmetries.

4.2.1 General Relativity as the Gauge Fixing $\phi \approx 0$

Before we show the existence of the dual theory we show that ADM general relativity can be recovered with the gauge fixing $\phi \approx 0$. With the addition of this gauge fixing condition to the constraint set, the constraints are no longer all first-class with respect to each other. The only constraint which contains π, and hence does not vanish when bracketed with ϕ, is the conformal constraint $T\mathcal{C}(\rho)$. Since,

$$[T\mathcal{C}(\rho), \phi] = [\pi, \phi] = \rho \not\approx 0, \tag{4.15}$$

the conformal constraint is second class (and by definition ϕ is too). Next we fix $\rho = 0$ in order that the first- and second-class constraints are preserved in time, then set the second-class constraints – ρ and $T\mathcal{C}$ – strongly equal to zero to ensure we remain on the constraint surface. (Note the propagation of constraints and elimination of second-class constraints appear similar but are two distinct steps.) We thus require

$$\phi = 0 \quad \text{and} \quad T\mathcal{C} = \pi - 4(p - \langle p\rangle\sqrt{g}) = 0$$

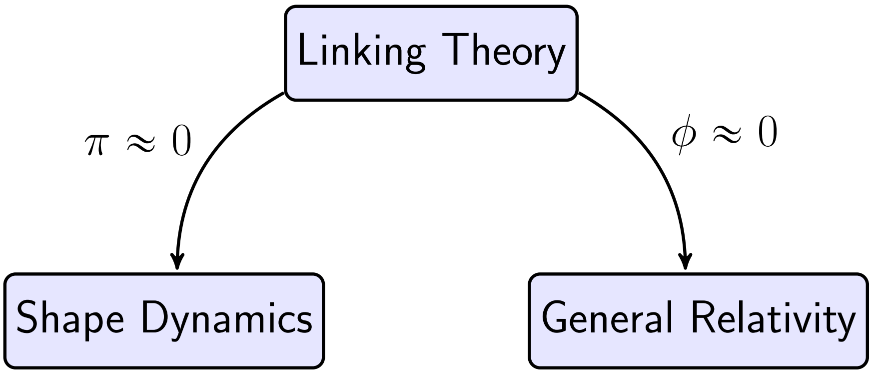

Figure 4.1: Shape dynamics and general relativity emerge from canonical gauge fixings of the linking theory.

or $\pi = 4(p - \langle p \rangle \sqrt{g})$. Since ρ is constrained to vanish by eq. (4.15), we are free to set $\pi = 0$ eliminating it from the theory altogether. It is not hard to see that the Hamiltonian constraint $T\mathcal{H}$ reduces to $\mathcal{H}$ on putting $\phi = 0$.

By the phase space reduction, $(\phi, \pi) = (0, 0)$, we obtain the original ADM phase space with the same constraints and have therefore recovered general relativity.

4.2.2 Shape Dynamics as the Gauge Fixing $\pi \approx 0$

The inclusion of the distinguished gauge fixing $\pi \approx 0$ in the constraint system results in the Hamiltonian constraint being demoted to second class status. To see explicitly we use the property that Poisson brackets are invariant under canonical transformations (see **??**) to rewrite $[T\mathcal{H}(N), \pi]$ as

$$T\left[\mathcal{H}(N), T^{-1}(\pi)\right] = T\left[\mathcal{H}(N), \pi + 4(p - \langle p \rangle \sqrt{g})\right] = 4T[\mathcal{H}(N), p - \langle p \rangle \sqrt{g}] \tag{4.16}$$

since $[\mathcal{H}(N), \pi] = 0$ and the inverse transform of π is $\pi + 4(p - \langle p \rangle \sqrt{g})$. It suffices to determine $[\mathcal{H}(N), p - \langle p \rangle \sqrt{g}]$:

$$[\mathcal{H}(N), p - \langle p \rangle \sqrt{g}] \approx 2\sqrt{g}\left(\nabla^2 - R - \frac{1}{4\sqrt{g}} p \langle p \rangle\right) N - \left\langle 2\sqrt{g}\left(\nabla^2 - R - \frac{1}{4\sqrt{g}} p \langle p \rangle\right) N \right\rangle . \tag{4.17}$$

Demanding that the above equals zero strongly gives an equation for N. This is called the *lapse fixing equation* and its solution ensures all constraints are propagated along trajectories. This equation is analogous to (4.15) in the previous section. Evidently, this step is now much more involved. Setting the second-class constraint π strongly equal to zero so that the conformal constraint becomes $p - \langle p \rangle \sqrt{g} \approx 0$ we can eliminate p from eq. (4.17). Defining the operator

$$\Delta := \nabla^2 - R - \frac{1}{4}\langle p \rangle^2$$

we rewrite the lapse-fixing equation as

$$\Delta N = \langle \Delta N \rangle . \tag{4.18}$$

We note the right-hand side is a constant. This is the *constant mean curvature* (CMC) lapse fixing equation which ensures the CMC foliation is preserved.

We do not need to solve (4.18) to realise there will be a family of solutions. This is because a solution N_1 to $\Delta N = c$, where c is an arbitrary constant on the hypersurface, will also be a solution to (4.18). (The spatial mean of a constant on hypersurfaces is of course the constant itself i.e. $\langle c \rangle = c$.) For a different c we obtain another solution N_2 which also satisfies (4.18), therefore, we have not completely fixed the lapse but retain a residual (one-parameter) freedom. This is in fact a non-trivial result and a crucial step in the construction. We denote a solution to (4.18) as N_0. Recall that second-class constraints are associated with fixed Lagrange multipliers, so the fact we have not eliminated the arbitrariness in N but retain a single parameter freedom suggests we still have a single first-class constraint left over. This constraint is the *global Hamiltonian*,

$$\mathcal{H}_{\text{SD}} := T\mathcal{H}(N_0) = \int \mathrm{d}^3x\, N_0\, T\mathcal{H}(x), \tag{4.19}$$

and is the generator of evolution in shape dynamics.[2] As a result, this is no longer a function of x. Nevertheless, it still has an associated arbitrary Lagrange multiplier which we simply multiply $\mathcal{H}_{\text{SD}}$ by – just as in the case of point mechanics.

Eq. (4.19) is invariant under conformal transformations and diffeomorphisms by virtue of being first-class with respect to the $T\mathcal{C}$ and $T\mathcal{H}^a$ constraints. This is not surprising; N_0 is the particular smearing which puts eq. (4.17) to zero. It marks a significant departure from general relativity as we have lost relative simultaneity for a single Hamiltonian. On the other hand if N was to be totally fixed, no Hamiltonian in $\int \mathrm{d}^3x\, N\, \mathcal{H}(x)$, of which there are infinitely many, would survive the second-class elimination: they are now all second-class. We would have no generator of dynamics, or, 'frozen' dynamics.
We now have a system with consistent dynamics generated by $\mathcal{H}_{\text{SD}}$. We can separate out this first-class $\mathcal{H}_{\text{SD}}$ from the rest of the (infinitely many) local Hamiltonians $T\mathcal{H}(x)$ by writing

$$\mathcal{H}_{\text{sc}}(x) := T\mathcal{H}(x) - \mathcal{H}_{\text{SD}}, \qquad x \in \Sigma_t.$$

Once we have split this first-class part out from the second-class constraints we set the $\mathcal{H}_{\text{sc}}$ strongly equal to zero by solving these constraints for $\hat{\phi}$ in terms of the original phase space variables g_{ab} and p^{ab}.

Now that we have identified all second-class constraints, we eliminate them by setting them strongly equal to zero to ensure that trajectories on the constraint surface remain on it. In the case of general relativity this was significantly easier since we were solving for the simple constraint $T\mathcal{C}$ for π. In shape dynamics we are solving the more complicated $T\mathcal{H}$ for $\hat{\phi}$ (not ϕ since we want volume preserving transformations). First, we simplify this constraint by applying $\pi = 0$ to $T\mathcal{C}$ to obtain the reduced conformal constraint

$$\mathcal{C}' := p - \langle p \rangle \sqrt{g} \approx 0$$

or $p \approx \langle p \rangle \sqrt{g}$. We then use this to eliminate the p's in $T\mathcal{H}[\hat{\phi}] = 0$. Defining the shape dynamic momenta $\sigma^{ab} = p^{ab} - \frac{1}{3} g^{ab} \langle p \rangle \sqrt{g}$, the equation $T\mathcal{H}[\hat{\phi}] = 0$ becomes

$$\frac{e^{-6\hat{\phi}}}{\sqrt{g}} \left(\sigma^{ab}\sigma_{ab} - \frac{1}{6}\langle p \rangle^2 g \right) - e^{8\hat{\phi}} \sqrt{g} R' = 0 \tag{4.20}$$

This is the so-called *Lichnerowicz-York equation.* It is used for constructing initial data in general relativity and a detailed examination of the existence and uniqueness properties can

[2] The generator is global because it involves the smearing over space.

be found in [13]. It suffices here to note that a solution, $\hat{\phi}_0 = \hat{\phi}_0(g_{ab}, p^{ab})$, exists when σ^{ab} is transverse ($\nabla_b \sigma^{ab} = 0$) and traceless ($\sigma^a{}_a = 0$) – this happens to be the case in shape dynamics [**?**]

Thus, the construction of shape dynamics is complete and we have traded refoliation invariance for conformal invariance and therefore have not gained or lost any physical degrees of freedom.

To summarise, the shape dynamics total Hamiltonian is

$$\mathcal{H}_{\text{tot}} = \mathcal{N}\mathcal{H}_{\text{SD}} + T\mathcal{H}^a(\xi_a) + \mathcal{C}'(\sigma) \tag{4.21}$$

the first term represents the unique evolution, the second represents spatial diffeomorphisms, and the last represents the local conformal transformations. (Note $\mathcal{N}$ is a (spatial) constant and not a function of x and by construction, $\mathcal{H}_{\text{tot}}$ is invariant under these gauge symmetries.)

4.2.3 CMC and Maximal Slicing Gauges

The form of the constraint $\mathcal{C}$ depends on group of conformal transformations we allow. In the unrestricted case (i.e., any conformal transformation) we have $\mathcal{C} = \pi - 4p \approx 0$, so carrying out the phase space reduction by demanding that $\pi = 0$ the constraint reads $\mathcal{C}' = p \approx 0$. Recalling that $p^{ab} = -\sqrt{g}(K^{ab} - g^{ab}K)$ or $p = 2\sqrt{g}K \approx 0$, where K is the mean extrinsic curvature, we see this is nothing but the *maximal slicing gauge*: $K = 0$. The restriction to the smaller group of volume preserving conformal transformations however, gives the constraint $\mathcal{C} = \pi - 4(p - \langle p \rangle \sqrt{g}) \approx 0$. Again imposing $\pi = 0$ we find $p - \langle p \rangle \sqrt{g} \approx 0$, i.e., the CMC gauge: $K = \langle p \rangle \sqrt{g} = \text{const}$.

4.3 Asymptotically Flat Shape Dynamics

4.3.1 Schwarzschild's solution in Shape Dynamics

Following the discovery of general relativity in 1915 the first solution to Einstein's field equations was given in 1916 by Schwarzschild for a spherically symmetric spacetime in vacuum. Likewise, the shape dynamics counterpart was published soon after its development in [5]. This solution shows that general relativity and shape dynamics can not be considered as equivalent theories with different gauge symmetries.

We will now outline the construction principle of a spherically symmetric solution and show that in the spacetime picture it forms a degenerate metric (i.e., non-invertible) as opposed to a non-degenerate metric in general relativity.

The Schwarzschild solution describes the spacetime around an isolated object. In the limit of large r, the spacetime can be considered flat. Thus, in shape dynamics the spatial hypersurfaces are asymptotically flat: $g_{ab} \to \eta_{ab}$ as $r \to \infty$. It does not make sense to request volume preserving transformations since space is now infinite and accordingly, we relax the restriction of volume preserving conformal transformations. However, we now need boundary conditions and so we specify the following: $N \to \mathcal{O}(1)$, $N^a \to \mathcal{O}(1)$, and $p^{ab} \to \mathcal{O}(r^{-2})$ as $r \to \infty$. The generating functional is the same as before but the equations are much tidier since expression will no longer involve averages $\langle \cdot \rangle$ here. The canonical variables in the linking theory are

$$g_{ab} \to e^{4\phi} g_{ab} \qquad p^{ab} \to e^{-4\phi} p^{ab},$$

and

$$\phi \to \phi \qquad \pi \to \pi - 4p.$$

The conformal constraint can be immediately written down: $\mathcal{C} = \pi \to T\mathcal{C} = \pi - 4p$. The diffeomorphism, Hamiltonian and conformal constraints in the linking theory are now

$$\begin{cases} \mathcal{H}^a \to T\mathcal{H}^a = -2e^{-4\phi}\nabla_b p^{ab} + 4e^{-4\phi} p \nabla^a \phi, \\ \mathcal{H} \to T\mathcal{H} = \frac{e^{-6\phi}}{\sqrt{g}}(p^{ab}p_{ab} - \frac{1}{2}p^2) - \sqrt{g}e^{2\phi}(R - 8\nabla^2\phi - 8\nabla^a\phi\nabla_a\phi) \approx 0, \\ \mathcal{C} \to T\mathcal{C} = \pi - 4p. \end{cases} \tag{4.22}$$

Using the shape dynamics gauge-fixing $\pi \approx 0$, the conformal constraint becomes $\mathcal{C}' := 4p$, i.e., the maximal-slicing gauge: $p = K = 0$. Now we can eliminate p in $T\mathcal{H}$ to find

$$T\mathcal{H} = \frac{e^{-6\phi}}{\sqrt{g}} p^{ab}p_{ab} - \sqrt{g}e^{2\phi}(R - 8\nabla^2\phi - 8\nabla^a\phi\nabla_a\phi) \approx 0. \tag{4.23}$$

Under this gauge-fixing, the only non-vanishing Poisson-bracket is with $T\mathcal{H}$:

$$[T\mathcal{H}(N), \pi] \approx -8N\frac{e^{-6\phi}}{\sqrt{g}}p^{ab}p_{ab} + 8\sqrt{g}e^{2\phi}(\nabla^2 N + 2\nabla^a\phi\nabla_a N) \approx 0$$

or

$$(\nabla^2 N + 2\nabla^a\phi\nabla_a N) - N\frac{e^{-8\phi}}{g}p^{ab}p_{ab} \approx 0. \tag{4.24}$$

(See appendix B.2 for details.) This is the lapse-fixing equation for asymptotically flat shape dynamics (c.f. eq. (4.17)) and we denote the solution N_0.

This completes the phase space reduction of the linking theory to shape dynamics and we are left with the set of first-class constraints:

$$\begin{cases} T\mathcal{H}^a(\xi_a) = \pounds_{\boldsymbol{\xi}} g_{ab}, \\ \mathcal{H}_{\text{SD}} = \int \mathrm{d}^3x\, N_0 \frac{e^{-6\phi}}{\sqrt{g}} p^{ab}p_{ab} - N_0\sqrt{g}e^{2\phi}(R - 8\nabla^2\phi - 8\nabla^a\phi\nabla_a\phi) \approx 0, \\ C' = 4p. \end{cases} \tag{4.25}$$

4.3.2 Solving the System

The equation of motion for g_{ab} is given by

$$\begin{aligned} \dot{g}_{ab} &= [g_{ab}, \mathcal{H}_{\text{tot}}] \\ &= [g_{ab}, \mathcal{H}_{\text{SD}}] + [g_{ab}, T\mathcal{H}^a(\xi_a)] + [g_{ab}, \mathcal{C}(\sigma)] \\ &= 2N_0\frac{e^{-6\phi}}{\sqrt{g}}p_{ab} + \quad \pounds_{\boldsymbol{\xi}} g_{ab} \quad + \quad 4\sigma g_{ab} \end{aligned} \tag{4.26}$$

In a spherically symmetric space, the metric is conformally flat meaning that it can be written as $g_{ab} = \Omega^4\eta_{ab}$. Eq. (4.26) then becomes

$$4\Omega^3\dot{\Omega}\eta_{ab} = 2N_0\frac{e^{-6\phi}}{\Omega^6\sqrt{\eta}}p_{ab} + \pounds_{\boldsymbol{\xi}}(\Omega^4\boldsymbol{\eta})_{ab} + 4\sigma\Omega^4\eta_{ab}. \tag{4.27}$$

In spherical coordinates spatial indices now run over r, ϕ, θ with $\boldsymbol{\eta} = \mathrm{diag}(1, r^2, r^2\sin\theta)$. Let us now make the simplifying assumption that the arbitrary vector field, $\boldsymbol{\xi}$ is directed radially outward: $\boldsymbol{\xi} = \xi(r,t)\partial/\partial r$. With this assumption in place, the Lie derivative can be written as (see appendix B.2 for technical details)

$$\pounds_{\boldsymbol{\xi}}\eta_{ab} = 2r\xi\left(\delta^{\theta}{}_{a}\delta^{\theta}{}_{b} + \delta^{\phi}{}_{a}\delta^{\phi}{}_{b}\sin^2\theta\right) + 2\delta^{r}{}_{a}\delta^{r}{}_{b}\xi_{,r}, \tag{4.28}$$

where $\delta^a{}_b$ is the delta-function in spherical coordinates. The first term of (4.28) can be rewritten in a more simple form:

$$(\eta_{ab} - \delta^{r}{}_{a}\delta^{r}{}_{b})\frac{2\xi}{r}$$

so (**??**) becomes

$$(\pounds_{\boldsymbol{\xi}}\Omega^4\boldsymbol{\eta})_{ab} = \Omega^4\left((\eta_{ab} - \delta^{r}{}_{a}\delta^{r}{}_{b})\frac{2\xi}{r} + 2\delta^{r}{}_{a}\delta^{r}{}_{b}\xi_{,r}\right) + \eta_{ab}\xi\Omega^4_{,r}. \tag{4.29}$$

Substituting this into $\dot{g}_{ab}$, **??**, and rearranging we obtain the following equation:

$$\left(4\Omega^3\dot{\Omega} - 4\xi\Omega^3\Omega_{,r} - 2\xi\frac{\Omega^4}{r} - 4\sigma\Omega^4\right)\eta_{ab} = \Omega^4(2\xi_{,r} - \frac{2}{r}\xi)\delta^{r}{}_{a}\delta^{r}{}_{b} + 2N_0\frac{e^{-6\phi}}{\Omega^6\sqrt{\eta}}p_{ab}. \tag{4.30}$$

We choose the arbitrary function $\sigma(t)$, to make the left-hand side bracket vanish at $t = 0$. Upon taking the trace of the remaining equation and recalling $p = 0$ we find

$$0 = 2\xi_{,r} - \frac{2}{r}\xi, \tag{4.31}$$

since $p = 0$ when we are in the maximal-slicing gauge. Thus, we must have

$$\xi(r,t) = cr$$

for some spatial constant $c = c(t)$. However, from the boundary conditions, $\xi(r,t) \to 0$ as $r \to \infty$, hence $c = 0$ so and

$$\boldsymbol{\xi} = 0.$$

With $\xi = 0$, (4.30) now reduces to

$$\left(4\Omega^3\dot{\Omega} - 4\sigma\Omega^4\right)\eta_{ab} = 2N_0\frac{e^{-6\phi}}{\Omega^6\sqrt{\eta}}p_{ab}. \tag{4.32}$$

The off-diagonal entries of η_{ab} are zero so we must then have $p_{ab} = 0$ at $t = 0$. Eq. (4.32) gives

$$4\Omega^3\dot{\Omega} - 4\sigma\Omega^4 = 0,$$

which is easily solved to give $\sigma = \dot{\Omega}/\Omega$.

Since we are constructing the vacuum solution, we have $R_{ab} = 0$. Using this and the condition $p_{ab} = 0$ at $t = 0$, the Hamiltonian constraint becomes

$$\nabla^a\nabla_a\phi + \nabla^a\phi\nabla_a\phi = 0. \tag{4.33}$$

Now setting $\phi = \ln\Omega$ this equation reduces to

$$\nabla^2\Omega = 0$$

which we recognise as Laplace's equation. It has a solution of the form

$$\Omega = a + \frac{b}{r}$$

where a and b may be functions of t. However, the boundary condition $\Omega = e^{\phi} \to 1 + \mathcal{O}(r^{-1})$ fixes $a = 1$ so

$$\Omega = 1 + \frac{b}{r}$$

Now we solve the lapse-fixing equation, eq. (4.24). Since we found $p^{ab} = 0$ at $t = 0$ the lapse fixing equation becomes

$$\nabla^2 N + 2\nabla^a \phi \nabla_a N = 0.$$

Substituting $\phi = \ln \Omega = \ln(1 + \frac{b}{r})$, this is an ordinary differential equation in r and has a solution satisfying the boundary conditions,

$$N = 1 - \frac{2b}{m + 2r}.$$

To show that the 3-metric is static and prove 'Birkhoff's theorem' we require that eq. (4.26) consists of only gauge generators, i.e., we need the dynamical term $2N\frac{e^{-6\phi}}{\sqrt{g}}p_{ab}$ to vanish. The lapse can not identically be zero so we have that p^{ab} must vanish. We use Gomes' reasoning that $\dot{p}^{ab} = 0$ at $t = 0$ implies $p^{ab} = 0$ for all t. This is not assured by the boundary condition, $p^{ab} \to \mathcal{O}(r^{-2})$. Further to this, a static geometry will have $(\dot{g}_{ab}, \dot{p}^{ab}) = (0, 0)$ so if we evaluate $\dot{p}^{ab}$ for each component with ϕ and N given by above, we find that the integration constant is $b = m$. In reconstructing the 3-metric, the pseudo-Riemannian geometry of spacetime dictates that the conformal factor Ω is set to one so that $g_{ab} = \eta_{ab}$. From the 3+1 split of spacetime, we know the 4-metric has the general form (in local coordinates)

$$\mathrm{d}s^2 = (-N^2 + \xi_a \xi^a)\,\mathrm{d}t^2 + 2\xi_a\,\mathrm{d}t\,\mathrm{d}x^a + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b.$$

The cross-term vanishes with spherical symmetry. Indeed, we found $\xi^a = 0$. Thus, we obtain

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{b}{r}}{1 + \frac{b}{r}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{b}{r}\right)^4 (\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2)).$$

This represents a wormhole in isotropic coordinates. When $r > b$ this is the exterior Schwarzschild solution.

Chapter 5

Discussion

In this report we have explored and reproduced the construction of a 'dual' theory of general relativity called shape dynamics as outlined in [1, 9]. Shape dynamics is a conformally invariant Hamiltonian description of gravity. is a conformally invariant theory with Although both theories are based in ADM phase space and maintain the same number of physical gravitational degrees of freedom, they possess different gauge symmetries. From the linking theory, there are two distinguished gauge fixings, each determining distinct theories. This is the basis of symmetry trading procedure which allows one to effectively replace refoliation invariance in general relativity by local conformal invariance given by $g_{ab} \to e^{4\hat{\phi}} g_{ab}$. (Note however, both theories remain diffeomorphism invariant.) In doing so, we no longer have the freedom to slice spacetime arbitrarily. Instead, a specific foliation is singled out: the constant mean curvature (CMC) gauge. The infinitely many local Hamiltonians are replaced by a global Hamiltonian which distinguishes a single solution in *superspace*. From our introductory remarks, this uniqueness is a desirable trait in a Machian theory, but comes at the unfortunate cost of relativity of simultaneity.

When we implemented the phase space reduction $(g_{ab}, p^{ab}, \phi, \pi) \to (g_{ab}, p^{ab})$, we obtained the Lichnerowicz-York equation. Remarkably, this is – up to a canonical transformation – precisely the equation one needs to solve in York's method [13] of solving the initial value problem. In his method, York showed that by performing conformal transformations, initial data could be constructed to satisfy the ADM constraints in the CMC gauge. However, the use of conformal methods by York was not well motivated but is one of the most reliable ways of constructing initial data in general relativity [14]. In this way, shape dynamics provides physical justification of York's conformal method.

The restriction to CMC foliable spacetimes only, means that solutions in shape dynamics may not have a counterpart in general relativity. As a result, solutions which satisfy Einstein's field equations may not be globally CMC sliceable, and likewise, not all solutions in shape dynamics will allow a non-degenerate 4-dimensional metric to be recovered. Although we may have patches of CMC and thus local equivalence of solutions, these are to be regarded as separate theories that have their own solutions. From the point of view of shape dynamics, solutions in general relativity that do not possess a CMC slicing are viewed as unphysical as a result. This is the consequence of having a theory that does not assume the existence of a spacetime but takes as its fundamental quantities, 3-dimensional conformal geometries, that is to say, the evolution of shape represents the true dynamics.

As a first application of shape dynamics, we followed [5] to show that shape dynamics admits a Birkhoff-type theorem – namely a theorem that specifies the unique vacuum solution to the Einstein equations in the case of spherical symmetry. The crucial step in showing that the 3-geometry, g_{ab}, is static relies on the momentum, p^{ab}, vanishing at all t. This is because the generator responsible for dynamics appears as a function of p^{ab}. Thus, if this generator is eliminated the 3-metric is static. In any proof of Birkhoff's theorem, this is the key step – to show that $\dot{g}_{ab}$ contains only gauge terms.

The fact that the Schwarzschild solution does not admit a maximal slicing, prevents an invertible metric from being reconstructed. Nevertheless, the degenerate metric is a wormhole in isotropic coordinates. In the exterior, the wormhole is the Schwarzschild solution, however this equivalence is not global, precisely because the Schwarzschild solution has no maximal slicing. This example shows how shape dynamics and general relativity are not equivalent.

Shape dynamics also avoids the *problem of time* found in quantum gravity. It arises because solutions to the ADM equations of motion are not unique – a different choice of lapse corresponds to the same physical solution, but a different curve in the superspace of all 3-metrics. On the other hand, a 'good' physical theory should be local; the global Hamiltonian in shape dynamics appears to violate this desire for locality. This issue disappears if the non-local theory can be gauge-fixed to a theory which can be considered local [10]. There is a caveat: the gauge-fixing conditions must be local for this 'definition' of locality. Shape dynamics fails in this regard, as the total gauge-fixing conditions are non-local in the Lagrange multiplier $\mathcal{N}$. More work is required here to understand the extent to which the shape dynamics can be mapped to a local theory via gauge-fixings.

The loss of refoliation invariance is significant, for we no longer have the problem of time often. Without the quadratic Hamiltonian constraint, the constraint algebra is much simpler. Future work, will largely focus on the quantum gravity aspects but shape dynamics, treated as a separate theory makes its own predictions and this is welcome regardless of its quantisation prospects.

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Appendix A

A.1 Canonical Transformations

Transformations of the form

$$q^i \to Q^i = Q^i(q,p,t)$$

$$p_i \to P_i = P_i(q,p,t)$$

are called *canonical transformations* if

$$[Q^i, P_j] = \delta^i{}_j. \tag{A.1}$$

Recall the Poisson bracket for two arbitrary functions $F(q,p)$ and $G(q,p)$

$$[F,G] = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}$$

using the chain rule and eq. (A.1) it can be easily shown that

$$[F,G] = \frac{\partial F}{\partial Q^i}\frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial Q^i}.$$

So,

$$[F,G]_{q,p} = [F,G]_{Q,P}.$$

that is, the Poisson bracket can be taken with respect to any pair of variables[1] defined by the Poisson bracket eq. (A.1) (subscripts denotes differentiation with respect to). Consequently,

$$\dot{Q}^i = [Q^i, H]_{q,p} = [Q^i, H]_{Q,P} = \frac{\partial H}{\partial P_i}$$

and likewise

$$\dot{P}_i = [P_i, H]_{q,p} = [P_i, H]_{Q,P} = -\frac{\partial H}{\partial P_i}$$

so a canonical transformation preserves the form of the canonical equations. [2]

Canonical transformations can be generated by generating function(al)s. There are four

[1] The distinction between coordinates and momenta is blurred under canonical transformations.
[2] In field theory we swap partial derivatives with functional derivatives.

types. In shape dynamics the generating functional is of the form $F = F(q, P)$ and relates the new and old canonical variables through the equations

$$p^{ab} = \frac{\delta F}{\delta g_{ab}}, \qquad G_{ab} = \frac{\delta F}{\delta P^{ab}}.$$

In shape dynamics the Poisson bracket is

$$[\mathcal{F}, \mathcal{G}] = \int \mathrm{d}^3x \left(\frac{\delta \mathcal{F}}{\delta g_{ab}} \frac{\delta \mathcal{G}}{\delta p^{ab}} - \frac{\delta \mathcal{F}}{\delta p^{ab}} \frac{\delta \mathcal{G}}{\delta g_{ab}} \right),$$

where $\mathcal{F}$ and $\mathcal{G}$ are functionals of g_{ab}, p^{ab}.

A.2 Action over Shape Space

Consider the simple 1-dimensional case. Let q be the coordinate and ϵ be the coordinate that specifies the "position" along the fibre (one can think of these as representing the gauge orbits). The action is

$$S = \int_{s_1}^{s_2} \mathrm{d}s \, L(q, \dot{q}, \epsilon, \dot{\epsilon}).$$

The aim is to extremize this with boundary values $q(s_1)$ and $q(s_2)$ specified as usual, but now we do not put any restrictions on the boundary values of ϵ because these are just gauge parameters and are free to vary. This amounts to just specifying an initial and final fibre. Varying the action with respect to ϵ as normal we have

$$\delta S = \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial \epsilon} - \frac{\mathrm{d}p}{\mathrm{d}s} \right) + \Big[p \, \delta\epsilon \Big]_{s_1}^{s_2} = 0, \tag{A.2}$$

where $p = \partial L / \partial \dot{\epsilon}$. Ordinarily the boundary term would vanish and we would recover the Euler-Lagrange equation, however, here in general it does not. It turns out that the boundary term does in fact vanish regardless and the argument goes as follows. The action must be stationary for all variations and in particular, for no variation at all i.e. for fixed endpoints. Thus the extremal trajectory must satisfy the Euler-Lagrange equation as a criterion. This is not sufficient however since this only fixes the trajectory for one kind of variation. Now consider the case when the initial end point is fixed but the final end-point is free. Because eq. (A.2) then differs from the Euler-Lagrange equation by

$$\Big[p \, \delta\epsilon \Big]_{s_1}^{s_2} = 0 - p \, \delta\epsilon \Big|_{s=s_1}$$

the value of the final end-point must also vanish in order to satisfy the Euler-Lagrange equation. Following the same argument with the initial end-point free and final end-point fixed we obtain the following condition that all variations must satisfy:

$$p(s_1) = p(s_2) = 0.$$

We can choose the initial and final fibres anywhere along the solution and the above condition will still hold so in fact we have $p(s) = 0$ everywhere along the curve. This a primary constraint which arises from the fact that we have some redundancy in configuration space when lifted from shape space. It should not be surprising there is a constraint as we must

compensate for adding an extra degree of freedom in ϕ. Geometrically this can be thought of as, curves – that satisfy the action principle – are 'orthogonal' with respect to the fibre bundle.
We can extend this to fields ($q^a \to \phi(x,t)$) in much the same way by replacing the partial derivatives with functional derivatives ($\partial/\partial\dot{q} \to \delta/\delta\dot{\phi}$).
Later we will see that this procedure is the reason for an additional constraint when we introduce gauge variables (playing the role of ϵ here) into the theory. (More details to be found in [3, 4].)

Appendix B

B.1 Technical Details

B.1.1 Useful variations

One variation we will need often is of the determinant $g := \det(g_{ab})$:

$$\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta g = \frac{1}{2}\sqrt{g}g^{ab}\delta g_{ab}.$$

To compute the variation in the Christoffel symbol we first vary

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^e{}_{cb}\, g_{ae} - \Gamma^e{}_{ca}\, g_{eb} = 0,$$

to find

$$\nabla_c \delta g_{ab} - g_{ae}\delta\Gamma^e{}_{cb} - g_{eb}\delta\Gamma^e{}_{ca} = 0,$$

and cycling through the indices we obtain three terms which we can combine to give the variation in the Christoffel symbol:

$$\delta\Gamma^c{}_{ab} = \frac{1}{2}g^{ce}(\nabla_a \delta g_{eb} + \nabla_b \delta g_{ea} - \nabla_c \delta g_{ab}).$$

The Ricci tensor becomes

$$\delta R = \delta(g^{ab}R_{ab}) = -R^{ab}\delta g_{ab} + g^{ab}\delta R_{ab},$$

where we have used $\delta g^{ab} = -g^{ac}g^{bd}\delta g_{cd}$. Now using Palatini's lemma:

$$\delta R_{ab} = \nabla_c \delta\Gamma^c{}_{ab} - \nabla_b \delta\Gamma^c_{ca}$$

we obtain

$$\delta R = -R^{ab}\delta g_{ab} + \nabla^a\nabla^b\delta g_{ab} - \nabla^2(g^{ab}\delta g_{ab}).$$

where $\nabla^2 := g_{ab}\nabla^a\nabla^b$.

Here we collect some useful results we will make further use of below. First recall $e^{4\hat{\phi}} =$

$e^{4\phi}\langle e^{6\phi}\rangle^{-2/3}$ where $\langle e^{6\phi}\rangle$ is a functional of g_{ab} so

$$\begin{aligned}
\frac{\delta e^{4\hat{\phi}(x)}}{\delta g_{ab}(y)} &= \left(\frac{\delta}{\delta g_{ab}}\langle e^{6\phi}\rangle^{-2/3}\right) e^{4\phi} \\
&= -\frac{2}{3}\langle e^{6\phi}\rangle^{-1} e^{4\hat{\phi}} \frac{\delta}{\delta g_{ab}}\left(V^{-1}\int \mathrm{d}^3x\sqrt{g}e^{6\phi}\right) \\
&= -\frac{2}{3}\langle e^{6\phi}\rangle^{-1} e^{4\hat{\phi}}\left(-V^{-2}\frac{\delta V}{\delta g_{ab}}\int \mathrm{d}^3x\sqrt{g}e^{6\phi} + V^{-1}\frac{1}{2}\sqrt{g}g^{ab}e^{6\phi}\right) \\
&= -\frac{2}{3}\langle e^{6\phi}\rangle^{-1} e^{4\hat{\phi}}\left(-\frac{1}{2}\sqrt{g}g^{ab}\langle e^{6\phi}\rangle V^{-1} + \frac{1}{2}\sqrt{g}g^{ab}V^{-1}e^{6\phi}\right) \\
&= \frac{1}{3}V^{-1}e^{4\hat{\phi}}(1-e^{6\hat{\phi}})\sqrt{g}g^{ab}
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\frac{\delta e^{4\hat{\phi}(x)}}{\delta \phi(y)} &= \langle e^{6\phi}\rangle^{-2/3}\frac{\delta}{\delta\phi}e^{4\phi} + e^{4\phi}\frac{\delta}{\delta\phi}\langle e^{6\phi}\rangle^{-2/3} \\
&= 4e^{4\hat{\phi}} - \frac{2}{3}\langle e^{6\phi}\rangle^{-5/3}e^{4\phi}V^{-1}\int \mathrm{d}^3x\sqrt{g}\frac{\delta e^{6\phi}}{\delta\phi} \\
&= 4e^{4\hat{\phi}} - \frac{2}{3}\langle e^{6\phi}\rangle^{-5/3}e^{4\phi}V^{-1}\int \mathrm{d}^3x\sqrt{g}\,6e^{6\phi}\delta(x,y) \\
&= 4e^{4\hat{\phi}} - 4\langle e^{6\phi}\rangle^{-1}e^{4\hat{\phi}}V^{-1}\sqrt{g}e^{6\phi} \\
&= 4e^{4\hat{\phi}(x)} - 4e^{4\hat{\phi}(x)}V^{-1}\sqrt{g}(y)e^{6\hat{\phi}(y)}
\end{aligned} \tag{B.2}$$

B.1.2 The Canonical Transformations

Recall for a general type-2 generating functional, $F = F(\phi_a, \Pi^a)$, for a canonical transformation $T:\ (\phi_a, \pi^a) \mapsto (\Phi_a, \Pi^a)$, the generating equations are of the form

$$\Phi_a = \frac{\delta F}{\delta \Pi^a} \quad \text{and} \quad \pi^a = \frac{\delta F}{\delta \phi_a}.$$

In shape dynamics the generating functional is

$$F = \int \mathrm{d}^3x\,(e^{4\hat{\phi}}g_{ab}P^{ab} + \phi\Pi) \tag{B.3}$$

and we have

$$\begin{aligned}
g_{ab} \;\to\; G_{ab} &= \frac{\delta F}{\delta P^{ab}} = e^{4\hat{\phi}}g_{ab} \\
\phi \;\to\; \Phi &= \frac{\delta F}{\delta \Pi} = \phi.
\end{aligned}$$

Computing the transformed momenta is more involved but using the results from above we find

$$p^{ab} = \frac{\delta F}{\delta g_{ab}} \quad \to \quad P^{ab} = e^{-4\hat{\phi}}\left(p^{ab} - \frac{1}{3}\langle p\rangle(1-e^{6\hat{\phi}})\sqrt{g}g^{ab}\right) \tag{B.4}$$

$$\pi = \frac{\delta F}{\delta \phi} \quad \to \quad \Pi = \pi - 4(p - \langle p\rangle\sqrt{g}) \tag{B.5}$$

where we have used eqs. (B.1) and (B.2) in inverting the above equations. We note that $\langle p\rangle$ is invariant under T:

$$\begin{aligned}
\langle p\rangle &= V^{-1}\int \mathrm{d}^3x\, p \\
&\to V^{-1}\int \mathrm{d}^3x \left(p - \langle p\rangle(1-e^{6\hat{\phi}})\sqrt{g}\right) \\
&= \left(\langle p\rangle - \langle p\rangle V^{-1}\int \mathrm{d}^3x\sqrt{g} + \langle p\rangle V^{-1}\int \mathrm{d}^3x\sqrt{g}e^{6\hat{\phi}}\right) \\
&= \left(\langle p\rangle - \langle p\rangle + \langle p\rangle\langle e^{6\hat{\phi}}\rangle\right) \\
&= \langle p\rangle
\end{aligned}$$

since volume is invariant under T and $\langle e^{6\hat{\phi}}\rangle = 1$ by eq. (4.10). Now $\pi - \langle p\rangle\sqrt{g}$ is also invariant under T. First observe

$$p \to Tg_{ab}p^{ab} = p - \langle p\rangle(1-e^{6\hat{\phi}})\sqrt{g}$$

and therefore

$$\begin{aligned}
T(p - \langle p\rangle\sqrt{g}) &= p - \langle p\rangle(1-e^{6\hat{\phi}})\sqrt{g} - \langle p\rangle e^{6\hat{\phi}}\sqrt{g} \\
&= p - \langle p\rangle\sqrt{g} + \langle p\rangle e^{6\hat{\phi}}\sqrt{g} - \langle p\rangle e^{6\hat{\phi}}\sqrt{g} \\
&= p - \langle p\rangle\sqrt{g}
\end{aligned}$$

which implies

$$T(\pi + 4(p - \langle p\rangle\sqrt{g})) = \pi,$$

indeed

$$T^{-1}\pi = \pi + 4(p - \langle p\rangle\sqrt{g}),$$

which we made use of in eq. (4.16) for deriving the lapse-fixing equation.

B.1.3 Constraints in the linking theory

Recall the diffeomorphism constraint:

$$\mathcal{H}^a = -2\nabla_b p^{ab} = -2(p^{ab}{}_{,b} + \Gamma^a{}_{bc}p^{cb} + \Gamma^b{}_{bc}p^{ac} - \Gamma^c{}_{bc}p^{ab}), \tag{B.6}$$

since we note that p^{ab} is a tensor density. Under the canonical transformation $g_{ab} \to e^{4\hat{\phi}}g_{ab} = \langle e^{6\phi}\rangle^{-2/3}e^{4\phi}g_{ab}$, the Christoffel symbol

$$\begin{aligned}
\Gamma^c{}_{ab} &\to \frac{1}{2}e^{-4\hat{\phi}}g^{cd}\left((e^{4\hat{\phi}}g_{ad})_{,b} + (e^{4\hat{\phi}}g_{db})_{,a} - (e^{4\hat{\phi}}g_{ab})_{,d}\right) \\
&= \frac{1}{2}e^{-4\phi}\langle e^{6\phi}\rangle^{2/3}g^{cd}\left(\langle e^{6\phi}\rangle^{-2/3}(e^{4\phi}g_{ad})_{,b} + \langle e^{6\phi}\rangle^{-2/3}(e^{4\phi}g_{db})_{,a} - \langle e^{6\phi}\rangle^{-2/3}(e^{4\phi}g_{ab})_{,d}\right) \\
&= \frac{1}{2}e^{-4\phi}g^{cd}\left((e^{4\phi}g_{ad})_{,b} + (e^{4\phi}g_{db})_{,a} - (e^{4\phi}g_{ab})_{,d}\right) \\
&= \Gamma^c{}_{ab} + 4\delta^c{}_{(a}\phi_{,b)} - 2g_{ab}g^{cd}\phi_{,d}
\end{aligned} \tag{B.7}$$

which gives the 3-Ricci scalar

$$R = 2g^{ab}(\Gamma^c{}_{a[b,c]} + \Gamma^c{}_{d[c}\Gamma^d{}_{b]a}) \to 8e^{-4\hat{\phi}}\left(\frac{1}{8}R - (\nabla\phi)^2 - \nabla^2\phi\right) \tag{B.8}$$

where $(\nabla\phi)^2 = \nabla^a\phi\,\nabla_a\phi$ and $\nabla^2\phi = \nabla^a\nabla_a\phi$. Now

$$\begin{aligned}
\Gamma^a{}_{bc}p^{bc} &\rightarrow \Gamma^a{}_{bc}p^{bc} - \frac{1}{3}\langle p\rangle(1-e^{6\hat{\phi}})\Gamma^a{}_{bc}\sqrt{g}g^{bc} \\
&\qquad\qquad +4p^{ab}\phi_{,b} + \frac{2}{3}\sqrt{g}g^{ab}\langle p\rangle(1-e^{6\hat{\phi}})\phi_{,b} - 2g^{ab}p\phi_{,b} \\
\Gamma^b{}_{bc}p^{ac} &\rightarrow \Gamma^b{}_{bc}p^{ac} - \frac{1}{3}\langle p\rangle(1-e^{6\hat{\phi}})\Gamma^b{}_{bc}\sqrt{g}g^{ac} \\
&\qquad\qquad +6p^{ab}\phi_{,b} - 2\sqrt{g}g^{ab}\langle p\rangle(1-e^{6\hat{\phi}})\phi_{,b} \\
\Gamma^c{}_{bc}p^{ab} &\rightarrow \Gamma^c{}_{bc}p^{ab} - \frac{1}{3}\langle p\rangle(1-e^{6\hat{\phi}})\Gamma^c{}_{bc}\sqrt{g}g^{ab} \\
&\qquad\qquad +6p^{ab}\phi_{,b} - 2\sqrt{g}g^{ab}\langle p\rangle(1-e^{6\hat{\phi}})\phi_{,b} \\
p^{ab}{}_{,b} &\rightarrow e^{-4\hat{\phi}}\bigg(p^{ab}{}_{,b} - 4p^{ab}\phi_{,b} + \frac{2}{3}\langle p\rangle\sqrt{g}g^{ab}e^{6\phi}\phi_{,b} - \frac{1}{3}\langle p\rangle(1-e^{6\phi})(\sqrt{g}g^{ab})_{,b} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{4}{3}\langle p\rangle\sqrt{g}g^{ab}\phi_{,b}\bigg)
\end{aligned}$$

so combining the above and noting that the tensor density, $\nabla_b(\sqrt{g}g^{ab}) = 0$, eq. (B.6) becomes

$$\mathcal{H}^a \quad\rightarrow\quad T\mathcal{H}^a = -2e^{-4\hat{\phi}}\Big[\nabla_b p^{ab} - 2(p - \langle p\rangle\sqrt{g})\nabla^a\phi\Big].$$

Using the constraint $T\mathcal{C} = \pi - 4(p - \langle p\rangle\sqrt{g}) \approx 0$ it can be written more compactly as

$$T\mathcal{H}^a \approx -2\nabla_b p^{ab} + \pi\nabla^a\phi. \tag{B.9}$$

Smearing this with an arbitrary vector field, $\boldsymbol{\xi}$, on hypersurfaces:

$$\begin{aligned}
T\mathcal{H}^a(\xi_a) &= \int_\Sigma \mathrm{d}^3x\,(-2\xi_a\nabla_b p^{ab} + \pi\xi_a\nabla^a\phi) \\
&= \int_\Sigma \mathrm{d}^3x\,\big(-2\nabla_b(\xi_a p^{ab}) + 2p^{ab}\nabla_b\xi_a + \pi\pounds_{\boldsymbol{\xi}}\,\phi\big) \qquad \text{(int. by parts)} \\
&= \int_\Sigma \mathrm{d}^3x\,\big(p^{ab}\nabla_a\xi_b + p^{ab}\nabla_b\xi_a + \pi\pounds_{\boldsymbol{\xi}}\,\phi\big) \\
&= \int_\Sigma \mathrm{d}^3x\,\big(p^{ab}\pounds_{\boldsymbol{\xi}}\,g_{ab} + \pi\pounds_{\boldsymbol{\xi}}\,\phi\big)\,, \qquad \text{(since } \pounds_{\boldsymbol{\xi}}\,g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a)
\end{aligned}$$

and we have assumed the vector field ξ^a vanishes asymptotically. Now recall during phase space reduction we impose $\pi = 0$. Clearly the last term vanishes and we recover the ADM diffeomorphism symmetry. The Lie derivative terms indicate that $T\mathcal{H}^a$ generate diffeomorphisms in Γ_{SD}. We confirm that $T\mathcal{H}^a(\xi_a)|_{\pi=0}$ indeed generates diffeomorphisms (of the 3-metric) in shape dynamics. Taking the Poisson bracket with the 3-metric:

$$\begin{aligned}
\delta g_{ab} &= [g_{ab}, T\mathcal{H}^a(\xi_a)] \\
&= \int \mathrm{d}^3x\,\delta^{cd}{}_{ab}\delta(x,y)(\pounds_{\boldsymbol{\xi}}g)_{cd}(x) = \pounds_{\boldsymbol{\xi}}g_{ab}
\end{aligned}$$

which is the infinitesimal diffeomorphism $g_{ab} \to g_{ab} + \pounds_{\boldsymbol{\xi}} g_{ab}$ on hypersurfaces Σ_t.
Using the above calculations we can immediately write down (with some algebra) the transformed Hamiltonian constraint:

$$\mathcal{H} \quad \to \quad T\mathcal{H} = \frac{e^{-6\hat{\phi}}}{\sqrt{g}} \left(p^{ab} p_{ab} - \frac{1}{2} p^2 + \frac{1}{3} p \langle p \rangle \sqrt{g} (1 - e^{6\hat{\phi}}) - \frac{1}{6} \langle p \rangle^2 g (1 - e^{6\hat{\phi}})^2 \right) \tag{B.10}$$

$$-8\sqrt{g} e^{2\hat{\phi}} \left(\frac{1}{8} R - (\nabla \phi)^2 - \nabla^2 \phi \right). \tag{B.11}$$

We show that $\mathcal{C}' = p - \langle p \rangle \sqrt{g}$ (SD conformal constraint) indeed generates conformal transformations. Smearing with an arbitrary σ:

$$\begin{aligned}
[g_{ab}, \mathcal{C}'(\sigma)] &= \int \mathrm{d}^3 x \, \frac{\delta g_{ab}}{\delta g_{cd}} \frac{\delta T\mathcal{C}(\sigma)}{\delta p^{cd}} \\
&= \int \mathrm{d}^3 x \, \delta(x, y) \left(\sigma(x) g_{ab}(y) - V^{-1} \left(\int \mathrm{d}^3 x' \sqrt{g} \sigma(x') \right) g_{ab}(y) \right) \\
&= \int \mathrm{d}^3 x \, \delta(x, y) \Big(\sigma(x) g_{ab}(y) - \langle \sigma \rangle g_{ab}(y) \Big) \\
&= (\sigma - \langle \sigma \rangle) g_{ab}
\end{aligned}$$

B.1.4 Verifying Second-Class Constraints in Shape Dynamics

Recall the smeared Hamiltonian constraint:

$$\mathcal{H}(N) = \int \mathrm{d}^3 x \, N \left[\frac{1}{\sqrt{g}} \left(p_{ab} p^{ab} - \frac{1}{2} p^2 \right) - \sqrt{g} R \right].$$

The following are needed to evaluate $[\mathcal{H}(N), p]$, showing that it does not vanish:

$$
\begin{aligned}
\frac{\delta \mathcal{H}(N)}{\delta g_{ab}} &= \frac{\delta}{\delta g_{ab}}\left(\frac{1}{\sqrt{g}}\right)\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) + \frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ab}}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) \\
&\qquad - \frac{\delta\sqrt{g}}{\delta g_{ab}}R - \sqrt{g}\frac{\delta R}{\delta g_{ab}} \\
&= \frac{1}{2\sqrt{g}}g^{ab}\left(p_{cd}p^{cd} - \frac{1}{2}p^2\right) + \frac{2}{\sqrt{g}}\left(g_{cd}p^{ac}p^{db} - \frac{1}{2}pp^{ab}\right) \\
&\qquad - \frac{1}{2\sqrt{g}}g^{ab}R - \sqrt{g}\left(-R^{ab}N + \nabla^a\nabla^b N - \nabla^2(g^{ab}N)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta \mathcal{H}(N)}{\delta p^{ab}} &= \frac{N}{\sqrt{g}}\frac{\delta}{\delta p^{ab}}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) \\
&= \frac{N}{\sqrt{g}}\left(2g_{ac}\, g_{bd}p^{cd} - g_{ab}p\right)
\end{aligned}
$$

$$
\frac{\delta p}{\delta p^{ab}} = g_{ab}
$$

$$
\frac{\delta p}{\delta g_{ab}} = p^{ab}
$$

So

$$
\begin{aligned}
\frac{\delta \mathcal{H}(N)}{\delta g_{ab}}\frac{\delta p}{\delta p^{ab}} &= -\frac{3}{2\sqrt{g}}(p_{ab}p^{ab} - p^2/2) + \frac{2}{\sqrt{g}}(g_{ac}\, g_{bd}p^{cd}p^{ab} - p^2/2)N \\
&\qquad - \frac{3}{2}\sqrt{g}RN - \sqrt{g}(-R + \nabla^2 N - 3\nabla^2 N) \\
&= \frac{3}{2}\left(\sqrt{g}R - \frac{1}{\sqrt{g}}(p_{ab}p^{ab} - p^2/2)\right)N + \frac{2}{\sqrt{g}}(p_{ab}p^{ab} - p^2/2)N \\
&\qquad - 2\sqrt{g}RN + 2\sqrt{g}\nabla^2 N \\
&= 2\sqrt{g}(\nabla^2 - R)N + \frac{2}{\sqrt{g}}(p_{ab}p^{ab} - p^2/2) - \frac{3}{2}NS \\
&\approx 2\sqrt{g}(\nabla^2 - R)N + \frac{2}{\sqrt{g}}(p_{ab}p^{ab} - p^2/2)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\delta \mathcal{H}(N)}{\delta p^{ab}}\frac{\delta p}{\delta g_{ab}} &= \frac{N}{\sqrt{g}}\left(2g_{ac}\, g_{bd}p^{ab}p^{cd} - g_{ab}p^{ab}p\right) \\
&= \frac{2N}{\sqrt{g}}(p_{ab}p^{ab} - p^2/2)
\end{aligned}
$$

and finally subtracting the last two expressions we find

$$
[\mathcal{H}(N), p] \approx 2\sqrt{g}(\nabla^2 - R)N \tag{B.12}
$$

We have outlined the asymptotically flat case in deriving the lapse fixing equation above. In the case of compact without boundary hypersurfaces the Poisson bracket required to vanish has an additional $-\langle p\rangle\sqrt{g}$ term:

$$[\mathcal{H}(N), p - \langle p\rangle\sqrt{g}] \approx 2\sqrt{g}\left(\nabla^2 - R - \frac{1}{4\sqrt{g}}p\langle p\rangle\right)N - \left\langle 2\sqrt{g}\left(\nabla^2 - R - \frac{1}{4\sqrt{g}}p\langle p\rangle\right)N\right\rangle$$

B.2 Asymptotically Flat Shape Dynamics

In this case, the scalar $\phi(x)$ is unrestricted and leads to much simpler canonical variables. Using the same generating functional eq. (B.3) we find

$$\begin{aligned} g_{ab} &\to e^{4\phi} g_{ab} \\ p^{ab} &\to e^{-4\phi} p^{ab} \end{aligned}$$

Since

$$\pi = \frac{\delta F}{\delta \phi} = \int \mathrm{d}^3x \, \left(4\delta(x,y)e^{4\phi}g_{ab}P^{ab} + \delta(x,y)\Pi\right) = 4p + \Pi$$

we have

$$\pi \to \Pi - 4p,$$

c.f. eqs. (B.4) and (B.5). It is not difficult to see that $T(p_{ab}p^{ab}) = p_{ab}p^{ab}$ and $T(p^2) = p^2$. The Christoffel symbols are independent of the scale factor i.e. whether we use ϕ or $\hat{\phi}$ so eq. (B.7) is valid here as well. As a result, we can reuse eq. (B.8) for the 3-Ricci scalar. With these considerations, the Hamiltonian constraint under the canonical transformation is

$$T\mathcal{H} = \frac{e^{-6\phi}}{\sqrt{g}}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) - 8e^{2\phi}\sqrt{g}\left(\frac{1}{8}R - \nabla^2\phi - (\nabla\phi)^2\right) \approx 0 \tag{B.13}$$

which is considerably simpler than eq. (B.10).
Moving on, under the gauge-fixing $\pi \approx 0$, the only non-vanishing Poisson bracket is

$$\begin{aligned} [T\mathcal{H}(N), \pi] &= \int \mathrm{d}^3x \frac{\delta T\mathcal{H}(N)}{\delta\phi(y)}\frac{\delta\pi(x)}{\delta\pi(y)} \\ &= -6\frac{e^{-6\phi}}{\sqrt{g}}p^{ab}p_{ab}N - 2\sqrt{g}e^{2\phi}(R - 8\nabla^2\phi - 8\nabla^a\phi\nabla_a\phi)N \\ &\qquad\qquad - \sqrt{g}e^{2\phi}(-8\nabla^2 N - 16\nabla^a\phi\nabla_a N) \\ &\approx -6N\frac{e^{-6\phi}}{\sqrt{g}}p^{ab}p_{ab} - 2N\frac{e^{-6\phi}}{\sqrt{g}}p^{ab}p_{ab} + 8\sqrt{g}e^{2\phi}(\nabla^2 N + 2\nabla^a\phi\nabla_a N) \\ &= -8N\frac{e^{-6\phi}}{\sqrt{g}}p^{ab}p_{ab} + 8\sqrt{g}e^{2\phi}(\nabla^2 N + 2\nabla^a\phi\nabla_a N) \end{aligned} \tag{B.14}$$

where we have weakly eliminated R by using $T\mathcal{H} \approx 0$ (note we could have equivalently computed $[\mathcal{H}(N), T^{-1}\pi]$). Demanding that eq. (B.14) is zero gives the lapse fixing equation

$$(\nabla^2 N + 2\nabla^a\phi\nabla_a N) - N\frac{e^{-8\phi}}{g}p^{ab}p_{ab} = 0. \tag{B.15}$$

Supposing now we have found the solution N_0 to the lapse fixing equation, we proceed to calculate the bracket $[g_{ab}, T\mathcal{H}(N_0)]$:

$$\begin{aligned}
[g_{ab}, T\mathcal{H}(N_0)] &= \int \mathrm{d}^3x\, \delta(x,y)\delta^{cd}{}_{ab}\frac{\delta}{\delta p^{cd}}\left(N_0\frac{e^{-6\phi}}{\sqrt{g}}p_{de}p^{de}\right)\\
&= 2N_0\frac{e^{-6\phi}}{\sqrt{g}}p_{ab}.
\end{aligned}$$

The gauge terms of $\dot{g}_{ab}$ can easily be calculated.

Here we show that the Lie derivative in the following form:

$$\begin{aligned}
\pounds_{\boldsymbol{\xi}}\eta_{ab} &= \xi\delta^c{}_r\nabla_c\eta_{ab} + \eta_{bc}\nabla_a\xi^c + \eta_{ac}\nabla_b\xi^c\\
&= \xi\left(\delta^r{}_a\delta^r{}_b\nabla_r\eta_{rr} + \delta^\theta{}_a\delta^\theta{}_b\nabla_r\eta_{\theta\theta} + \delta^\phi{}_a\delta^\phi{}_b\nabla_r\eta_{\phi\phi}\right) + \eta_{bc}\nabla_a\xi\delta^c{}_r + \eta_{ac}\nabla_b\xi\delta^c{}_r\\
&= \xi\left(\delta^\theta{}_a\delta^\theta{}_b\nabla_r\eta_{\theta\theta} + \delta^\phi{}_a\delta^\phi{}_b\nabla_r\eta_{\phi\phi}\right) + \eta_{br}\nabla_a\xi + \eta_{ar}\nabla_b\xi\\
&= \xi\left(\delta^\theta{}_a\delta^\theta{}_b\nabla_r\eta_{\theta\theta} + \delta^\phi{}_a\delta^\phi{}_b\nabla_r\eta_{\phi\phi}\right) + \delta^r{}_a\delta^r{}_b\nabla_r\xi + \delta^r{}_a\delta^r{}_b\nabla_r\xi\\
&= \xi\left(\delta^\theta{}_a\delta^\theta{}_b\nabla_r\eta_{\theta\theta} + \delta^\phi{}_a\delta^\phi{}_b\nabla_r\eta_{\phi\phi}\right) + 2\delta^r{}_a\delta^r{}_b\partial_r\xi\\
&= \xi\left(\delta^\theta{}_a\delta^\theta{}_b 2r + \delta^\phi{}_a\delta^\phi{}_b 2r\sin^2\theta\right) + 2\delta^r{}_a\delta^r{}_b\partial_r\xi\\
&= 2r\xi\left(\delta^\theta{}_a\delta^\theta{}_b + \delta^\phi{}_a\delta^\phi{}_b\sin^2\theta\right) + 2\delta^r{}_a\delta^r{}_b\partial_r\xi
\end{aligned}$$